Finite Point Method for Convection-Dominated Flow Problems

Chinapat Buachart and Worsak Kanok-Nukulchai

Abstract— *In this paper, the new upwinding scheme incorporate to meshfree technique has been proposed to overcome the instability issues in convection-dominated flow problem. This techniques is then demonstrated in one and two-dimensional problems using meshfree point collocation method. Numerical results for example problems demonstrate the techniques developed in this paper are effective to solve convection dominated problems.*

Keywords— **Meshfree, point collocation, upwinding, convection-flow.**

1. INTRODUCTION

Many practical problems in engineering are governed by the so-called convection-diffusion equations, in which, there are both convective and diffusive terms. To analyze the convection-diffusion problems, the conventional finite element method (FEM), the finite difference method (FDM), or the finite volume method (FVM) has been widely used.

However, there is a well-known instability issue of convection-diffusion problem; the highly oscillatory solution will occurs when the convective term is dominated. Hence, the special treatment is required to stabilize the numerical solution. A lot of studies have been performed to solve the instability problem in conventional numerical schemes, and excellent documentation on stabilization techniques can be found in the book written by Zienkiewicz and Taylor [1].

In recent years, meshfree or meshfree methods have attracted more and more attention from many researchers in computational mechanics field, especially in computational fluid mechanics. These meshfree methods do not required a mesh to discretize the problem domain, because the shape function is constructed entirely based on a set of scattered nodes. They are categorized to be two groups. First category is based on the collocation techniques, for example the finite point method (FPM) [2] and the *hp*-cloud method [3]. Other meshfree methods are based on weak form, e.g. the element-free Galerkin (EFG) method [4], the reproducing kernel particle (RKP) method [5], the meshfree local Petrov-Galerkin (MLPG) method [6], and so on, this category of meshfree method require numerical integration to approximate the weak form. Hence, the first category is defined to be truly meshfree, *i.e.* no mesh required to perform numerical integration.

Only very few studies on solving convection dominated problems using meshfree method have been reported. Oñate *et al.* [2] applied the finite point method to the convection dominated problem with upwinding for the first derivative or with characteristics approximation. Atluri *et al.* [6] used the MLPG method with local upwinding weight to solve the convection-diffusion problems. Gu and Liu [7] proposed several adaptive support domain techniques incorporate with meshfree collocation method to solve convection-dominated problem successfully.

In this paper, techniques to stabilize the convection dominated problems are developed and investigated for finite point method (FPM). These techniques were developed based on edge stabilization with analytical solution of equivalent one-dimensional convection dominated equation. Numerical results demonstrate that using these techniques, the instability problem caused by convection term can be solved effectively via meshfree method.

2. APPROXIMATION IN FPM

Weighted-Least Square Approximation

Let suppose Ω_i is an local approximation domain (cloud) of unknown function $u(\boldsymbol{x})$ consists of *star point* $\boldsymbol{x}_i$, and set of points $\boldsymbol{x}_j$, $j = 2,3,\ldots,np$ surrounding star point as shown in Figure 1.

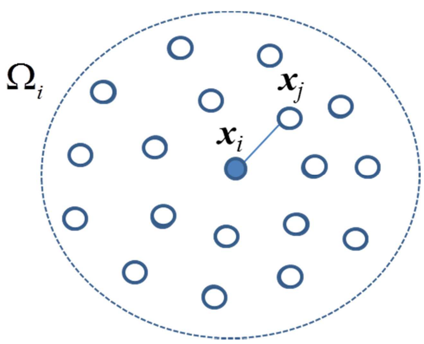

Fig.1. Local Approximation Domain in WLS.

Then, the local approximation of u can be defined by

$$\hat{u} = \sum_{l=1}^{m} p_l\left(\boldsymbol{x}\right)\alpha_l = \boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{x}\right)\boldsymbol{\alpha} \tag{1}$$

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where $\boldsymbol{p}(\boldsymbol{x})$ is a vector whose m-components are the terms of a complete polynomial basis functions and $\boldsymbol{\alpha}$ is a vector of coefficients which must be determined. In this work, complete quadratic polynomial bases are employed, e.g. $\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x})$ = [1, x, y, x^2, xy, y^2] in 2-dimensional domain. Note that the point's coordinates are relative to the star point position. For each region Ω_i the number of could nodes is larger than number of parameter $\boldsymbol{\alpha}$, $np > m$. To solve for approximation parameter $\boldsymbol{\alpha}$, we define the functional

$$J_i = \sum_{j=1}^{np} \varphi_i\left(\boldsymbol{x}_j\right)\left[\hat{u}_j - u_j\right]^2 \tag{2}$$

in which $\varphi_i(\boldsymbol{x}_j) = \varphi(\boldsymbol{x}_j - \boldsymbol{x}_i)$ is a compact support weighting function centered on the star point $\boldsymbol{x}_i$ and np is the number of points in the local cloud. This procedure, known as fixed Weighted Least-Squares (WLS), can be considered as a particular case of the Moving Least-Square (MLS) method proposed by Lancaster and Salkauskas in [8]. The minimization of Eq. (2) with respect to parameter $\boldsymbol{\alpha}$ leads to the following system of equations

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{B}\boldsymbol{u} \tag{3}$$

where $\mathbf{A} = \mathbf{P}^{\mathrm{T}}\boldsymbol{\Phi}(x)\mathbf{P}$, $\mathbf{B} = \mathbf{P}^{\mathrm{T}}\boldsymbol{\Phi}(x)$, $\boldsymbol{\Phi}(x) = \mathrm{diag}[\varphi_i(\boldsymbol{x}_j)]$, and $\mathbf{P}^{\mathrm{T}} = [\boldsymbol{p}(\boldsymbol{x}_1), \boldsymbol{p}(\boldsymbol{x}_2), \ldots, \boldsymbol{p}(\boldsymbol{x}_{np})]$. Then, the vector of unknown coefficient can be obtained by inversion of matrix $\mathbf{A}$, *i.e.* $\boldsymbol{\alpha} = \mathbf{A}^{-1}\mathbf{B}\mathbf{u}$, and the approximation value of $u(\boldsymbol{x})$ at the star point of the cloud is computed by Eq. (1)

$$\hat{u}_i = \boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{x}_i\right)\mathbf{A}^{-1}\mathbf{B}\boldsymbol{u} = \sum_{j=1}^{np} a_{ij}u_j \tag{4}$$

where a_{ij} is the approximation coefficient or shape function at star point $\boldsymbol{x}_i$. Having adopted a fixed weighting function, matrices $\mathbf{A}$ and $\mathbf{B}$ become constant in Ω_i, thus fist-order derivatives of the unknown function at star point are approximated in the FPM by

$$\frac{\partial \hat{u}_i}{\partial x_k} = \frac{\partial \boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{x}_i\right)}{\partial x_k}\mathbf{A}^{-1}\mathbf{B}\boldsymbol{u} = \sum_{j=1}^{np} b_{ij}^{k}u_j \tag{5}$$

in which b^k_{ij} is the first partial derivative of shape function a_{ij} with respect to x_k and higher-order derivatives can be obtained by successive differentiation of basis function vector. Henceforth, the WLS approximation of Laplacian operator can be computed by

$$\nabla^2\hat{u}_i = \nabla^2\boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{x}_i\right)\mathbf{A}^{-1}\mathbf{B}\boldsymbol{u} = \sum_{j=1}^{np} c_{ij}u_j \tag{6}$$

where c_{ij} is the Laplacian of shape function, defined by $c_{ij} = \nabla^2 a_{ij}$.

The Weighting Function

There exits many possibilities for choosing the functional form of a weighting function. In this paper the following normalized Gaussian (exponential) weighting function is adopted

$$\varphi_i\left(\boldsymbol{x}_j\right) = \frac{e^{-\left(wd_j/\gamma d_{\max}\right)^2} - e^{-w^2}}{1 - e^{-w^2}} \tag{7}$$

where d_j = distance between star point $\boldsymbol{x}_i$ and surrounding point $\boldsymbol{x}_j$ in cloud. The parameters w and γ govern the shape of the weighting function. In this work, we will set the value of γ = 1.01. Next, to assign the admissible range of parameter w, the effect of parameter w on the shape of Gaussian weight function is illustrated in Figure 2.

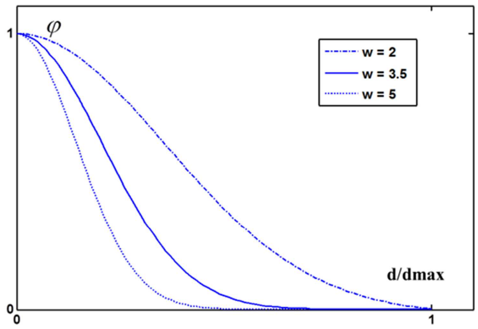

Fig. 2. Effects of the parameter *w* on the weighting function.

For large values of w, the shape function a_{ij} tends to the Dirac delta function (see Figure 2) and the approximation procedure tends to interpolate nodal data. Larger value w causes the error in the approximation to decrease, while the condition number of matrix $\mathbf{A}$ increased [9]. As a result, the system Equation (3) becomes more and more ill-conditioned. Hence, this parameter should be properly set. The maximum value of parameter w is set to be 3.5 in a whole domain and then it is reduced by factor 0.85 for each cloud of points whenever necessary, in order to obtain a given accuracy in the approximation or until reach the minimum value of 2.0. If the minimum value of parameter w were reached, then the influence domain will be increased by 25 percent, *i.e.* $d_{max}^{(\mathrm{new})} = 1.25\ d_{max}^{(\mathrm{old})}$, and the process to compute the proper value of w will be repeated again.

Consistency of Approximation

It is usual practice in meshfree method to associate the ability of approximation coefficients to reproduce a given polynomial of order p and its derivatives in an exact way. A set of approximation coefficients a_{ij}, b_{ij}^k and c_{ij} from WLS approximation have to satisfy following conditions

$$\sum_{j=1}^{np} a_{ij}\boldsymbol{p}_j = \boldsymbol{p}\left(\boldsymbol{x}_i\right)$$
$$\sum_{j=1}^{np} b_{ij}^{k}\boldsymbol{p}_j = \frac{\partial \boldsymbol{p}\left(\boldsymbol{x}_i\right)}{\partial x_k}$$
$$\sum_{j=1}^{np} c_{ij}\boldsymbol{p}_j = \nabla^2\boldsymbol{p}\left(\boldsymbol{x}_i\right) \tag{8}$$

where $\boldsymbol{p}(\boldsymbol{x})$ is a complete polynomial base of order p and $\boldsymbol{p}_j = \boldsymbol{p}(\boldsymbol{x}_j)$. Due to the fact that the weight function use to construct WLS approximation is fixed, the shape function and its derivatives are discontinuous. It is *only* possible to satisfy the consistency requirements (8) at the star point where the center of weighting function is located.

3. MODEL GOVERNING EQUATIONS

Two-Dimensional Convection-Diffusion Problem

Consider the advection-diffusion equation in two-dimensional space [10]:

$$\frac{\partial \phi}{\partial t} + \boldsymbol{u} \cdot \nabla \phi = \kappa \nabla^2 \phi + f \tag{9}$$

where $\phi(\boldsymbol{x}, t)$ is the dependent variable, a scalar-valued function of the spatial coordinates $\boldsymbol{x}$ and time t. The advection velocity field is denoted by $\boldsymbol{u}(\boldsymbol{x})$ and the positive coefficient $\kappa(\boldsymbol{x})$ represent diffusivity. The body source term denoted by $f(\boldsymbol{x},t)$.

We allow the essential and natural boundary conditions (diffusive flux), respectively:

$$\phi(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), \forall x \in \Gamma_g \tag{10}$$

$$\boldsymbol{n} \cdot \kappa \nabla \phi(\boldsymbol{x}, t) = h(\boldsymbol{x}, t), \forall x \in \Gamma_h \tag{11}$$

where $\boldsymbol{n}$ is the unit outward normal vector to the boundary $\Gamma = \Gamma_g \cup \Gamma_h$ and g and h are prescribed functions. The steady-state solution is defined by $\phi(\boldsymbol{x})$ when time derivative term in Equation (9) equals to zero.

4. THE MESHFREE FPM SOLVER

Spatial Discretization

The scalar variable ϕ and derivatives are approximated by the WLS scheme as described in section 2. Therefore, for each star point $\boldsymbol{x}_i$ we have the following numerical approximations

$$\begin{aligned} \hat{\phi}(\boldsymbol{x}_i) &= \hat{\phi}_i = \sum_{j=1}^{np} a_{ij} \phi_j^h \\ \frac{\partial \hat{\phi}(\boldsymbol{x}_i)}{\partial x_k} &= \frac{\partial \hat{\phi}_i}{\partial x_k} = \sum_{j=1}^{np} b_{ij}^k \phi_j^h \\ \nabla^2 \hat{\phi}(\boldsymbol{x}_i) &= \sum_{j=1}^{np} c_{ij} \phi_j^h \end{aligned} \tag{12}$$

It is importance to note that the nodal parameters ϕ^h do not coincide with the approximated ones $\hat{\phi}$ because in the WLS approximation the shape functions do not interpolate nodal data. These values are related by first line of Equation (12), which implies that a linear system have to be solved in order to obtain the nodal parameters ϕ^h. Experiment shows that this equation system can be solved by a few iterations of a Gauss-Seidel method or similar [11, 12]. Then, taking advantage of the partition of nullity (PN) property of the shape function derivatives, it is possible to infer

$$\sum_{j=1}^{np} b_{ij}^k = 0 \rightarrow b_{ii}^k = -\sum_{j \neq i} b_{ij}^k \tag{13}$$

and replacing Equation (13) into second line of Equation (12) the following expression for first derivative of scalar variable is obtained

$$\frac{\partial \hat{\phi}_i}{\partial x_k} = \sum_{j \neq i} b_{ij}^k \left(\phi_j^h - \phi_i^h \right) \tag{14}$$

However, Equation (14) is unstable and needs to be stabilized. For that purpose, a more suitable equivalent form is proposed [11, 12, 13], which is given by

$$\frac{\partial \hat{\phi}_i}{\partial x_k} = 2 \sum_{j \neq i} b_{ij}^k \left(\phi_{ij} - \phi_i^h \right) \tag{15}$$

where ϕ_{ij} is a mean value of scalar variable at the midpoint of the line segment connecting the star point $\boldsymbol{x}_i$ to another point $\boldsymbol{x}_j$ in cloud. This stabilized value ϕ_{ij} is calculated by concerning the equivalent one-dimensional exact solution of convection dominated equation. The stencil of points used in the calculation of Equation (15) is presented in Figure 3. The theoretical issue of stabilized variable will be described in the next section.

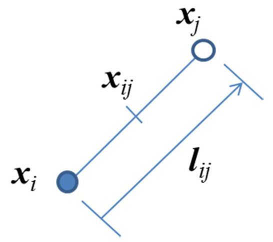

Fig. 3. The one-dimensional stencil of points.

Upwind Computation of First Derivative

The general solution of the steady-state equivalent one-dimensional convection-diffusion equation with no source term is represented by the exponential form [14]:

$$\phi = A + B e^{\alpha \xi} \tag{16}$$

where $\alpha = \boldsymbol{u} \cdot \boldsymbol{l}_{ij} / 2\kappa$ is the stencil Peclet number and the normalized coordinate $-1 \leq \xi \leq 1$. Substituting the essential boundary conditions at both ends, namely $\phi(-1) = \phi_i$ and $\phi(1) = \phi_j$, we obtain

$$2A = \phi_i + \phi_j - \left(\phi_j - \phi_i\right)\coth\alpha$$
$$2B\sinh\alpha = \left(\phi_j - \phi_i\right) \tag{17}$$

Hence, the mean value ϕ_{ij} can be expressed by average integral of solution in Equation (16) over the line segment as shown in Figure 3:

$$2\phi_{ij} = \phi_i + \phi_j - \zeta\left(\phi_j - \phi_i\right) \tag{18}$$

where ζ is the stencil *dissipation* coefficient, which its magnitude less than or equal to one. Using the value of coefficients *A* and *B* from Equation (17), we obtain the *optimal upwind scheme* (OU):

$$\zeta = \coth\alpha - \frac{1}{\alpha} \tag{19}$$

A simplified scheme which avoids the calculation of hyperbolic cotangents and maintains second-order accuracy [15] is given by

$$\begin{aligned} \zeta &= -1 - 1/\alpha, & \alpha < -1, \\ &= 0, & -1 \le \alpha \le 1, \\ &= 1 - 1/\alpha, & 1 < \alpha, \end{aligned} \tag{20}$$

We refer to Equation (19) as the *critical upwind* scheme (CU). Moreover, the most simple *full upwind scheme* (FU) defined by

$$\begin{aligned} \zeta &= -1, & \alpha < 0, \\ &= 0, & \alpha = 0, \\ &= 1, & \alpha > 0, \end{aligned} \tag{21}$$

which results in the first-order accuracy scheme. Thus, the approximation of convection term in governing equation (9) at star point *i* can be expressed as follow:

$$\boldsymbol{u}\cdot\nabla\hat{\phi}_i = u_1\frac{\partial\hat{\phi}_i}{\partial x_1} + u_2\frac{\partial\hat{\phi}_i}{\partial x_2} := 2\sum_{j\neq i} d_{ij}\left(\phi_{ij} - \phi_i^h\right) \tag{22}$$

in which $d_{ij} = u_1 b_j^1 + u_2 b_j^2$, the algebraic divergence operator. Further simplify of Equation (22) using the result from (18), obtain the stabilized convection flux (denoted by dF_i) as follow:

$$dF_i = \boldsymbol{u}\cdot\nabla\hat{\phi}_i := \sum_{j\neq i} d_{ij}\left(1-\zeta\right)\left(\phi_j^h - \phi_i^h\right) \tag{23}$$

The stabilized flux in expression (23) will be used to compute the convection term in next section

Discretization in Time

From the governing equation (9), expressions (12) of the WLS approximation procedure, and the discretization of the first derivatives, equations (14), (15) and (23) we obtain the following semi-discretization form for the unknown at each star point *i*:

$$\sum_{j=1}^{np} a_{ij}\frac{\partial\phi_j^h}{\partial t} = r_i \tag{24}$$

In which $r_i = -dF_i + \kappa\sum_{j=1}^{np} c_{ij}\phi_j^h + f_i$, represented the right-hand-side or residual term at current time. Integrate equation (24) in time using Euler scheme with critical time step size [11, 12] for each star point *i* leads to the incremental equation as follow

$$\sum_{j=1}^{np} a_{ij}\Delta\phi_j^h = \Delta t r_i \tag{25}$$

where $\Delta\phi_j^h = \phi_j^h\left(t+\Delta t\right) - \phi_j^h\left(t\right)$, the difference of nodal unknown parameter between current and next time step. Equation (24) can be solved iteratively by adding unknown parameters at star point *i* on both sides:

$$\Delta\phi_i^{h,(n+1)} = \Delta t r_i + \Delta\phi_i^{h,(n)} - \sum_{j=1}^{np} a_{ij}\Delta\phi_j^{h,(n)} \tag{26}$$

where *n* = 0, 1, 2, …; denotes number of iterations, and $\Delta\phi_j^{h,(0)} = 0$. The iterative form in Equation (26) typically converges within few iteration steps [12, 13].

5. NUMERICAL EXAMPLES

Numerical solutions presented here onwards have been obtained with a FPM based on WLS approximation, taken from literature [2, 16]. Half-size of critical time step is used in time integration process.

One-dimentional: Sinusoidal Source Term

The one-dimensional convection-diffusion problem with homogeneous *boundary condition* at both ends is investigated. Twenty-one equally spaced points have been used to discretize unit length domain. Convective velocity u = 1 m/s and diffusivity parameter κ = 0.01 m^2/s.

Figure 4 shows the steady-state results for equation (9) with a sinusoidal source term $f = \sin \pi x$ using three schemes, namely OU, CU and FU as described in previous section. The values of unknown parameters ϕ_i^h solved by optimal and critical upwind show no different and very close to analytical solution. A small over diffusion appears for FU scheme.

One-dimentional: Discontinuous Initial Profile

The transient one-dimensional convection-diffusion problem, which represents contaminant transport phenomena, is simulated following the detail in Yim and Mohsen [16]. The domain length is 200 ft. Convective velocity u = – 0.1 ft/day and diffusivity parameter κ = 0.01 ft^2/day. The numerical simulations are performed using time step equals to 0.2 day, incorporate with CU scheme in Equation (20) to compute stabilized convective flux.

Results from two sets of equally spaced grids are shown in this example. First grid, which is called *coarse* grid, discretized using hundred-one equally spaced points. Second grid has been more refined by twice. Figure 8 shows the results for t = 200, 400 and 600 days and their corresponded analytical solution from [16]. The over diffusion results are observed. However, the result from finer grid produces less diffusion phenomena. Note that all numerical results predict peak position absolutely correct.

Two-dimentional case

The next example is analysis of two-dimensional thermal convection-diffusion a square domain under a uniform heat source f = 10 using a regular square grid of 8 × 8 points with diagonal convective velocity $u_1 = u_2$ = 1 m/s and diffusive coefficient κ = 0.005 m^2/s. A prescribed zero value, *i.e.* homogeneous boundary condition, of the temperature at the boundary has been taken. Figures 5 and 6 show the results for the unknown parameters ϕ_i^h. The numerical solution is free of oscillations and coincides with the expected result [2] for all stabilized schemes. The slightly over diffusion solution from FU scheme can be clearly seen from Figure 7, which shown profile plot along diagonal line of two dimensional solutions from Figures 5 and 6.

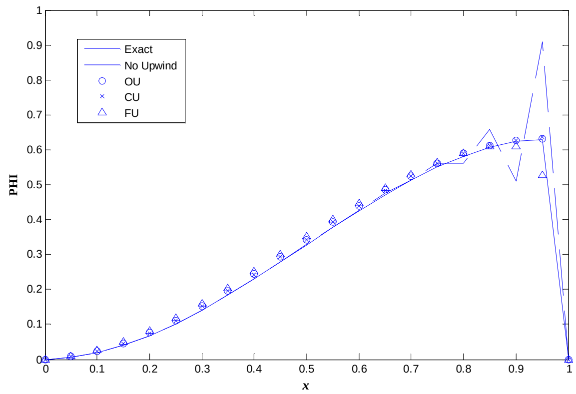

Fig. 4. One-dimensional solution with various stabilized schemes.

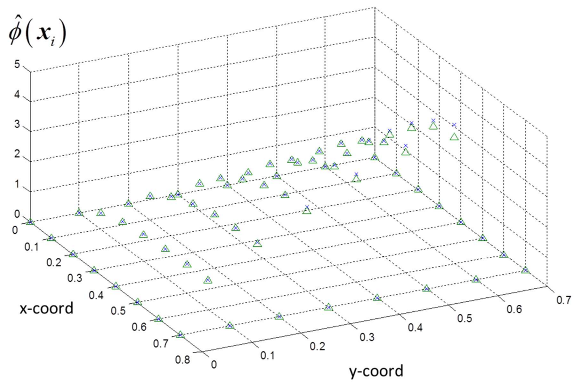

Fig. 5. Two-dimensional solution with diagonal velocity field.

Cross-point shows results from OU and CU schemes; Triangular marks represent FU result.

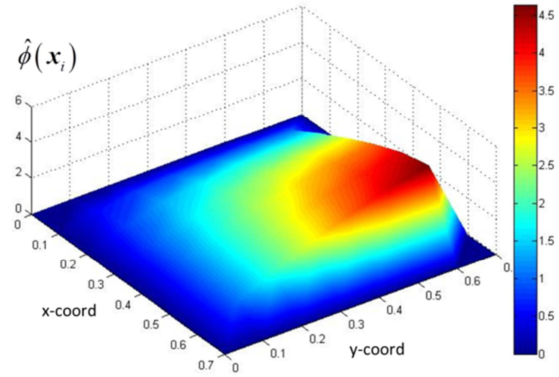

Fig. 6. Contour of two-dimensional solution with diagonal velocity field, the optimal upwind (OU) scheme.

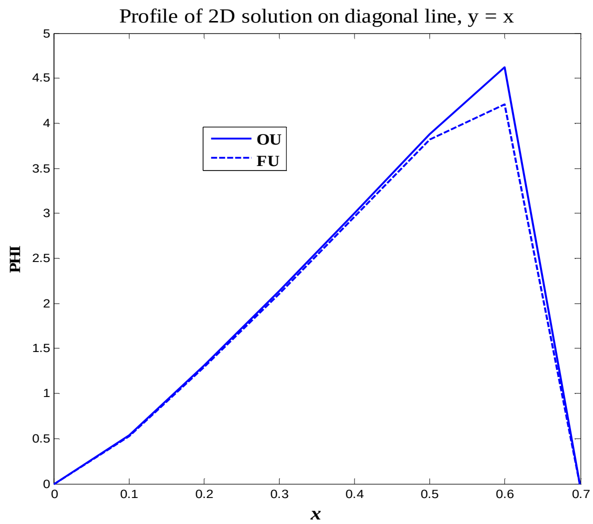

Fig.7. Profile of solution along diagonal line (y = x) for two-dimensional problem with diagonal velocity field.

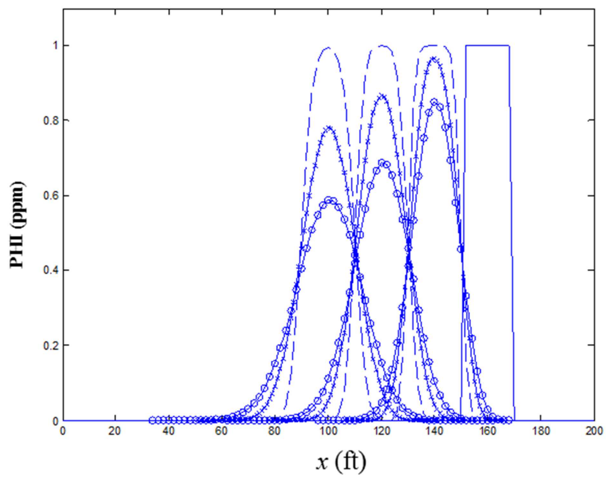

Fig. 8. Solution of transient 1D discontinuous initial profile (solid ine) at *t* = 200, 400 and 600 days. Dash line represent exact solutions, circle (o) and cross (x) points show coarse and fine grids result, respectively.

6. CONCLUSION

The finite point method (FPM) with upwind discretization of first derivatives to solve convection-dominated flow is presented. The advantage of the method compared with standard finite element method (FEM) is to avoid the necessity of mesh generation and compared with classical finite difference method (FDM) is the facility to handle the non-structured distribution of points.

The results from numerical examples show that present FPM are sufficiently accurate. Steady-state solutions are perfectly matched with analytical solution. Transient solutions with simple Euler time stepping scheme show over diffusion results. In this case, point refinement might be used to improve accuracy. Searching for method to reduce over diffusion should be interested research topic. Extension of this method to solve other specific fluid mechanic problems should be further investigated.

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