Improvement of Shielding Models with $h$-Conformal Formulation using a Subproblem Finite Element Technique

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Abstract— The aim of this work is to present an extension of the subproblem finite element approach to improve inaccuracies around curvatures and corners from thin plate finite element models. Traditional shielding magnetic models are generally replaced by surfaces with impedance-type interface conditions, which accept inaccuracies of the local solutions surrounding corners and borders such as magnetic fields, eddy current losses and Joule power losses. In this article, the extended subproblem approach is performed with the $h$-conformal finite element formulation where the shielding models are coupled in a sequence of a three step-coupling: Stranded inductors alone (step 1) – added shielding models (step 2) - volume corrections (step 3). Each step of the method allows separately solving on its own sub-domain and meshing without starting again a full problem for each new set of parameters. This leads to a reduced computational time.

Keywords— Eddy current, joule power loss, magnetodynamics, subproblem technique, thin shell approximations.

1. INTRODUCTION

The direct use of the finite element method (FEM) formulation to realistic thin plates in electromagnetic shielding problems is still extremely challenging, even not possible [1]. For magnetodynamic problems, when thin shells are conducting, very thin meshes have to be created to capture the skin depth, this becomes more and more computationally expensive (and difficult) with increasing frequency.

So as to scope with this advantage, in [2], [3], the classical thin shell (TS) solution is proposed to replace volume thin meshes (Fig. 1, left) by surface meshes with impedance-type interface conditions (ICs) (Fig. 1, right), but this approximation neglects significant errors in the calculation of local quantities (magnetic flux density, magnetic field, eddy current and joule power loss...) around corners and edges.

In this paper, a three step-coupling technique with the $h$-conformal magnetodynamic finite element (FE) formulation is extended from a one-way coupling to treat inaccuracies on local electromagnetic solutions from the TS approximations [2], [3]. The extended method is implemented as a sequence, allowing to divide a full problem (including of stranded inductors and magnetic thin regions) into three steps:

• Step 1: A problem involving stranded inductors is first considered on a simplified mesh.
• Step 2: A thin shell is added with very coarse mesh that does not contain stranded inductor anymore.
• Step 3: A volume correction with an actual thin plate is improved errors of the TS solutions.

In this strategy SP, from the stranded inductor to the TS model is constrained by surface sources (SSs) that indicate changes of ICs across surfaces [4]-[6]. From the TS model to the volume correction is constrained by SSs and volume sources (VSs), where VSs present changes of material properties (i.e., permeability and conductivity) of volume thin regions [4]-[6].

Each process of the method allows separately doing on its own domain and meshing without depending on the meshes and domains of other subproblems (SPs). This also permits to consider previous solutions as SSs or VSs for new SPs instead of solving a new full problem for each new set of parameters.

2. SEQUENCE OF SUBPROBLEM APPROACH

Magnetodynamic problem

In the scenario of the subproblem approach, a magnetodynamic problem $q$ is performed at step $q$ in a domain $\Omega_q$, with $\Omega_q = \Omega_{c,q} \cup \Omega_{s,q}$ and the boundary $\partial \Omega_q = \Gamma_{c,q} \cup \Gamma_{b,q}$. Where $\Omega_{c,q}$ and $\Omega_{s,q}$ are respectively the conducting part and the non-conducting. The Maxwell’s equations together with the following constitutive relations express [4] - [8].
\[
\text{curl } h_q = j_{s,q}, \text{div } b_q = 0, \text{curl } e_q = -\partial_t b_q, \tag{1a-b-c}
\]
\[
b_q = \mu h_q + b_{s,q}, e_q = \sigma_a^{-1} j_q + e_{s,q}, \tag{2a-b}
\]
\[
n \times e_q|_{\Gamma_q} = 0, [n \times e_q]_{\gamma_q} = k_{f,q}, \tag{3a-b}
\]
where the source fields \((b_{s,q} \text{ and } e_{s,q})\) in (2a-b) are VSSs and \(n\) is the unit normal exterior to \(\Omega_q\). The trace of electric field in (3a) is defined as a zero for classical homogeneous boundary conditions (BCs). The discontinuity of electric field in (3b) \((k_{f,q})\) is a SS expressed its discontinuity through the positive and negative sides \((\gamma_+^q \text{ and } \gamma_-^q)\) of any interface \(\gamma_q\) in \(\Omega_q\), with notation \([n]_{\gamma_p} = |\gamma_+^q| - |\gamma_-^q|\) [4-7].

**Sequence of SPs: From stranded inductor alone to TS model**

As developed in [4]-[7], the constraint from stranded inductor alone to the TS models is expressed a SS (i.e. \(k_{f,q}\) in (3b)). This SS linked to the BC and IC is presented by the TS model associated with contributions from the previous problem [2], [3]. For the magnetic field formulation, the magnetic field is split as \(h_q = h_{c,q} + h_{d,q}\), where the fields \(h_{c,q}\) and \(h_{d,q}\) are respectively continuous and discontinuous components of \(h_q\) across the TS [3].

At the step 1, there is no any thin region in the stranded inductor, but it should be assumed that there is a relative constraint with the TS model across the ICs of the TS \((q = p)\), with \(\gamma_q = \gamma_+^q = \gamma_-^q\). For that, the trace discontinuity \([n \times e_p]_{\gamma_p}\) in (3b) of the TS problem \(p\) is given as [3]

\[
[n \times e_p]_{\gamma_p} = [n \times (e_q + e_p)]_{\gamma_p} = [n \times e]_{\gamma_p} = \mu_p \beta_p \partial_t (2 h_{c,p} + h_{d,p}), \tag{4}
\]

\[
n \times e_{\gamma_p} = n \times (e_q + e_p)|_{\gamma_p} - n \times e_q|_{\gamma_p} = 1/2 \left[ \mu_p \beta_p \partial_t (2 h_{c,p} + h_{d,p}) + \frac{1}{\sigma_p} \beta_p h_{d,p} \right] - n \times e_q|_{\gamma_p} = 1/2 \left[ \mu_p \beta_p \partial_t (2 h_{c,p} + h_{d,p}) + \frac{1}{\sigma_p} \beta_p h_{d,p} \right] - k_{f,q}, \tag{5}
\]
where \(\beta_p\) is given in [4]. In the stranded inductor (e.g. SP q), the trace discontinuity \([n \times e_p]_{\gamma_p}\) in (4) is equal to zero (i.e. \([n \times e]_{\gamma_p} = [n \times e_q]_{\gamma_p} = [n \times e_q]_{\gamma_p} = 0\)) because it does not contribute to the solution of SP q [3].

**Sequence of SPs: From TS model to volume correction**

The TS solution of the problem SP \(p\) is next improved by the volume correction SP \(k\) \((q = k)\) that scopes with the TS approximations [3]. So as to improve the TS model, one must remove the TS representation solution appearing in the SP \(k\) by imposing a SS opposed to the TS IC. The changes of properties from the TS model to the volume correction are defined via the VSSs \((b_{s,k} \text{ and } e_{s,k})\) in (2a-b), i.e. [6], [9]

\[
b_{s,k} = (\mu_k - \mu_p)(h_q + h_p), \tag{6}
\]
\[
e_{s,k} = -(e_q + e_p), \tag{7}
\]
where the electric fields \(e_q\) and \(e_p\) are unknown and are defined via an electric problem [9].

### 3. SEQUENCE OF WEAK FORMULATIONS

From the results of Section 2, the weak formulations for each step corresponding to each SP are developed as the sequence: SP \(q\) → SP \(p\) → SP \(k\).

**Step 1: A weak form for inductors – SP q**

The magnetic field formulation is written from the weak form of Faraday’s equation (1 c) [5]-[10]. The field \(h_q\) is divided into two parts [6], [9], \(h_q = h_{r,q} + h_{s,q}\), where \(h_{r,q}\) is a reaction field and is unknown in advance, and \(h_{s,q}\) is a source field defined by a fixed current density such that \(\text{curl } h_{s,q} = j_{s,q}\) in the stranded inductor \(\Omega_{s,q}\). For the SP \(q\), one gets

\[
\partial_t (\mu_q h_{r,q} h^\prime_q)_{\Omega_q} + \partial_t (\mu_q h_{s,q} h^\prime_q)_{\Omega_q} + (n \times e_q, h^\prime_q)_{\Gamma_q} = 0, \forall h^\prime_q \in H^1_{\gamma_q}(\text{curl, } \Omega_q), \tag{8}
\]
where the field \(h_{s,q}\) is determined via a projection method of the unknown distribution \(j_{s,q}\) [5], [6]. \(H^1_{\gamma_q}(\text{curl, } \Omega_q)\) is a function space including the basis function \((h_{c,q})\) and the test function \((h^\prime_q)\) as well. The term \((n \times e_q, h^\prime_q)_{\Gamma_q} \text{ is considered as a natural BC given in (3 a).}\)

**Step 2: A weak form for TS model – SP p**

The weak form for the TS model \(p\) is expressed via the terms \([n \times e_p]_{\gamma_p}\) and \(n \times e_p|_{\gamma_p}\) given in (4) and (5), i.e.

\[
\partial_t (\mu_p h_p, h^\prime_p)_{\Omega_p} + (\sigma_p^{-1} \text{curl } h_p, h^\prime_p)_{\Omega_{c,p}} + (n \times e_p, h^\prime_p)_{\Gamma_{p-}} = 0, \forall h^\prime_p \in H^1_{\gamma_p}(\text{curl, } \Omega_p), \tag{9}
\]

The test function \(h^\prime_p\) in the trace discontinuity \((n \times e_p)_{\gamma_p}\) and \(n \times e_p|_{\gamma_p}\) in (10) is analysed as

\[
[(n \times e_p)_{\gamma_p}, h^\prime_p]_{\gamma_p} = [(n \times e_p)_{\gamma_p}, h^\prime_{c,p} + h^\prime_{d,p}]_{\gamma_p} = [(n \times e_p)_{\gamma_p}, h^\prime_{c,p}]_{\gamma_p} + [(n \times e_p)_{\gamma_p}, h^\prime_{d,p}]_{\gamma_p}, \tag{10}
\]

The term \((n \times e_p)_{\gamma_p}, h^\prime_{d,p})_{\gamma_p}\) is expressed as

\[
[(n \times e_p)_{\gamma_p}, h^\prime_{d,p}]_{\gamma_p} = (n \times e_p|_{\gamma_p}, h^\prime_{d,p})_{\gamma_p} + (n \times e_p|_{\gamma_p}, h^\prime_{d,p})_{\gamma_p}, \tag{11}
\]

where the \(h^\prime_{d,p}\) is equal to zero on \(\Gamma_0^-\) [3]. Therefore, (11) is rewritten

\[
[(n \times e_p)_{\gamma_p}, h^\prime_p]_{\gamma_p} = [(n \times e_p)_{\gamma_p}, h^\prime_{c,p}]_{\gamma_p} + (n \times e_p|_{\gamma_p}, h^\prime_{d,p})_{\gamma_p}, \tag{12}
\]
where the first integral \(\langle n \times e_p, h'_k \rangle_{\Gamma_p} \) and the second term \(\langle n \times e_p, h'_d \rangle_{\Gamma_p} \) in the SHS are respectively given in (4) and (5), i.e.

\[
\langle n \times e_p, h'_k \rangle_{\Gamma_p} = (\mu_p \beta_p \partial_t (2h_{c,p} + h_{d,p}), h'_k)_{\Gamma_p} + \frac{1}{2} \left( \frac{\mu_p \beta_p}{\sigma_p \beta_p} \partial_t (2h_{c,p} + h_{d,p}) + \frac{1}{\sigma_p \beta_p} h_{d,p} \right), h'_k)_{\Gamma_p} - k_{f,q}.
\]  
(13)

The \(k_{f,q} \) in (13) is a SS defined from the weak form of a previous problem SP \(q \), i.e. [6]

\[
-k_{f,q} = -\langle n \times e_q, h'_d \rangle_{\Gamma_p} = \partial_t(\mu_q h_{r,q} h'_q)_{\Omega_q} + \partial_t(\mu_q h_{s,q} h'_q)_{\Omega_q} = \Gamma_p \tag{14}
\]

By combining the equations from (11) to (15), the weak form of the TS model finally becomes

\[
\partial_t(\mu_p h_p, h'_p)_{\Omega_p} + (\mu_p \beta_p \partial_t (2h_{c,p} + h_{d,p}), h'_p)_{\Gamma_p} + \frac{1}{2} \left( \frac{\mu_p \beta_p}{\sigma_p \beta_p} \partial_t (2h_{c,p} + h_{d,p}) + \frac{1}{\sigma_p \beta_p} h_{d,p} \right), h'_p)_{\Gamma_p} + \partial_t(\mu_q h_{r,q} h'_q)_{\Omega_q} + \partial_t(\mu_q h_{s,q} h'_q)_{\Omega_q} = \Gamma_p \tag{14}
\]

\[
\langle n \times e_p, h'_p \rangle_{\Gamma_p} = 0, \forall h'_p \in H^1(\text{curl}, \Omega_p). \tag{15}
\]

At the discrete level, the SSs \(h_{r,q} \) and \(h_{s,q} \) initially defined the mesh of SP \(q \) are transferred to the mesh of the SP \(p \) through a projection method [11].

**Step 3: A weak form for volume correction – SP \(k \)**

The weak form for the volume correction SP \(k \) is presented as

\[
\partial_t(\mu_k h_k, h'_k)_{\Omega_k} + (\sigma_k^{-1} \text{curl} h_k, \text{curl} h'_k)_{\Omega_k} + \partial_t(\mu_k h_k, h'_k)_{\Omega_k} + (e_{s,k} \text{curl} h'_k)_{\Omega_k} + \langle n \times e_k, h'_k \rangle_{\Gamma_k} = 0, \forall h'_k \in H^1(\text{curl}, \Omega_k). \tag{16}
\]

By substituting VSs \(b_{s,k} \) in (6) and \(e_{s,k} \) in (7) into (17), one becomes

\[
\partial_t(\mu_k h_k, h'_k)_{\Omega_k} + (\sigma_k^{-1} \text{curl} h_k, \text{curl} h'_k)_{\Omega_k} + \partial_t((\mu_k - \mu_p)(h_q + h_p), h'_k)_{\Omega_k} + (- (e_q + e_p), \text{curl} h'_k)_{\Omega_k} + \langle n \times e_k, h'_k \rangle_{\Gamma_k} = 0, \forall h'_k \in H^1(\text{curl}, \Omega_k). \tag{17}
\]

where the fields \(h'_k \) in SP \(q \) and \(h'_d \) in SP \(p \) are projected from the meshes of the SP \(q \) and the SP \(p \) to the mesh of SP \(k \) through a projection method [11].

In addition, the representation of the TS model \(\langle n \times e_k, h'_k \rangle_{\Gamma_k} \) in (18) needs to be removed by a SS, i.e. [4-7]

\[
\langle n \times e_k, h'_k \rangle_{\Gamma_k} = -\langle n \times e_p, h'_k \rangle_{\Gamma_k}. \tag{18}
\]

It should be noted that the electric fields \(e_q \) and \(e_p \) in (17) are unknown and need to be defined via an electric problem presented in next Section.

### Weak formulation for electric problem

As mentioned in the previous Section, the source quantities \(e_q \) and \(e_p \) in (17) are found via an electric problem in \(\Omega_{c,i} \) \((i = q, p) \) [9]. The weak form for the electric problem for the volume correction SP \(k \) is obtained from the associated (1c), the unknown electric vector potential \(u_k \) \(\mathbf{d}_k = \text{curl} \mathbf{u}_k \) and the constitutive law \(\mathbf{d}_k = \varepsilon_k \mathbf{e}_k \), i.e.

\[
\partial_t(\mu_k h_k, u'_k)_{\Omega_k} + (e_k^{-1} \text{curl} \mathbf{u}_k, \text{curl} u'_k)_{\Omega_k} + \langle n \times e_k, u'_k \rangle_{\Gamma_k} = 0, \forall u'_k \in H^1(\text{curl}, \Omega_k). \tag{19}
\]

where \(H^1(\text{curl}, \Omega_k) \) is a function space defined in \(\Omega_k \) including the basis function \(u_k \) and test function \(u'_k \) as well. The expression for each test function \(u'_k \) can be directly written only for non-conducting region \(\Omega^c_k = \Omega_k \setminus \Omega_{c,k} \) i.e. [9]

\[
(\varepsilon_k^{-1} \text{curl} \mathbf{u}_k, \text{curl} u'_k)_{\Omega^c_k} + (\partial_t(\mu_k h_k), u'_k)_{\Omega^c_k} + (\sigma_k^{-1} \text{curl} \mathbf{u}_k, \text{curl} u'_k)_{\Omega^c_k} + \langle n \times e_k, u'_k \rangle_{\Gamma^c_k} = 0, \forall u'_k \in H^1(\text{curl}, \Omega_{c,k}). \tag{20}
\]

In the (20), the source field \(h_k \) is transferred from mesh of previous meshes to the mesh of \(\Omega_{c,k} \). The volume integrals in \(\Omega^c_k \setminus \Omega_{c,k} \) are defined in a single layer of FEs touching \(\partial \Omega_{c,k} \) in \(\Omega_k \setminus \Omega_{c,k} \). The solution \(u_k \) obtained from (20) is considered as a source for computing the electric field.

### 4. NUMERICAL TESTS

The first application problem is a TEAM workshop problem 28 [13] (Fig. 2). It consists of two stranded inductors and an above shielding plate, for \(\mu_{plate} = 100, \sigma_{plate} = 34 \text{ MS/m}, f = 50 \text{ Hz} \). The inner inductor has \(w_1 = 960 \) turns and the outer inductor \(w_2 = 576 \) turns. The sinusoidal currents flow in the inductors in opposite directions, i.e. \(i(t) = 20 \sin(2\pi ft) (A) \).

![Image](image.png)

**Fig. 2. Geometry 2-D of TEAM problem 28 (dimensions in mm) [14].**
The test problem is solved with three steps. The sequence of associated solutions on magnetic and electric fields of each SPs are shown in Figure 3. A problem with the stranded inductors alone SP $q$ (Fig. 3, top, $h_q$, inductors alone), added TS solution SP $p$ (second level, $h_p$), electric field for SP $q$ (third level, $e_q$), electric field for SP $p$ (fourth level, $e_p$) and volume correction SP $k$ (bottom, $h_k$). The colored map of eddy current density on the TS solution ($j_p$, top) and the volume correction ($j_k$, bottom) ($\mu_{plate} = 100$, $\sigma_{plate} = 34$ MS/m, $f = 50$ Hz).

Fig 5. Eddy current density (top) and joule power loss density (bottom) of the TS, improvement and reference solutions along the shielding plate for different thicknesses ($\mu_{plate} = 100$, $\sigma_{plate} = 34$ MS/m, $f = 50$ Hz).

The colored map of eddy current density on the TS solution and volume correction is pointed in Figure 4. The significant inaccuracy on the eddy current density along the plate of TS SP $p$ corrected by the volume improvement SP $k$ are shown in Figure 6. It reaches 64%, with $\delta = 1.22$mm and $d = 5$mm, or lower than 30%
for $d = 3$ mm. The improvement solution is then compared to be very similar the reference solutions obtained from the classical finite element method [1].

The second application is also based on a TEAM workshop problem 21 (model, B) [14]. The problem is solved in 3-D case and is performed as the same sequence of the previous test. The significant errors on the eddy current are depicted by colored maps with thickness $d = 1.5$ mm and $d = 7.5$ mm (from left to right) (Fig. 6). The relative improvement on the joule power loss along the quarter-plate, from the middle to the end, with the different thicknesses, is presented in Figure 7. It can be up to 60% near the end of the TS for $d = 7.5$ mm, and 40% with $d = 1.5$ mm, for $\mu_{\text{plate}} = 100, \sigma_{\text{plate}} = 6.484$ MS/m and $f = 50$ Hz in both cases.

![Colored map on the eddy current densities showing the regions with the relative volume improvements greater than 5% for $d = 1.5$ mm and $10$ mm (from left to right), $\mu_{\text{plate}} = 100, \sigma_{\text{plate}} = 6.484$ MS/m, $f = 50$ Hz.]

**Fig. 6.** Colored map on the eddy current densities showing the regions with the relative volume improvements greater than 5% for $d = 1.5$ mm and $10$ mm (from left to right), $\mu_{\text{plate}} = 100, \sigma_{\text{plate}} = 6.484$ MS/m, $f = 50$ Hz.

![Relative improvement of joule power loss (3D) along the quarter-plate, from the middle to the end, with different thicknesses $\mu_p = 200, \sigma_{\text{plate}} = 6.484$ MS/m, $f = 50$ Hz.]

**Fig 7.** Relative improvement of joule power loss (3D) along the quarter-plate, from the middle to the end, with different thicknesses $\mu_p = 200, \sigma_{\text{plate}} = 6.484$ MS/m, $f = 50$ Hz.

5. CONCLUSION

The subproblem FE technique has been successfully proposed with $k$-conformal formulation for improving the errors on the local quantities of the TS model in three steps. The sequence of each step is implemented on its own sub-domain without staring again a full domain. From this step to another one is contrained via VSs and SSs. In particular, an electric problem has been also introduced in the heart of method to strongly support the VSs in volume correction.

The method has been successfully applied to the international problems (TEAM workshop problem 21 (model B) and problem 28). The simulated results of the method are checked to be quite similar to the reference solution calculated from the finite element method [1].

This is a very good agreement of the studied method. The development has been done with the linear case in the frequency domain. It can be extended to the nonlinear case and time domain in the future work.

**NOMENCLATURE**

The list of symbols used in the paper is given below:

- $H_{\text{eq}}^2(\text{curl}, \Omega_q)$: Curl-conform function space in $\Omega_q$
- $(\cdot, \cdot)$: Volume integral of the product of its vector field argument
- $< \cdot, \cdot, \cdot >$: Surface integral of the product of its vector field argument
- $h_q$: Magnetic field (A/m)
- $b_q$: Magnetic flux density (T)
- $e_q$: Electric field (V/m)
- $d_q$: Electric flux density (C/m²)
- $f_{s,q}$: Electric current density (A/m²)
- $j_q$: Eddy current density (A/m²)
- $b_{s,q}, e_{s,q}$: Volume sources
- $k_{f,q}$: Surface source field
- $\Omega_q$: Bounded open set of $E^3$
- $\Gamma_q$: Boundary of $\Omega_q$ ($\Gamma_q = \partial \Omega_q$)
- $\mu$: Magnetic permeability (H/m)
- $\sigma$: Electric conductivity (S/m)
- $\rho$: Electric charge (C/ m³)

**REFERENCES**


